HOLOGRAPHIC RENORMALIZED ENTANGLEMENT AND ENTROPIC C-FUNCTION

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Based on MF-He-Sun, Phys. Rev. D102, 126019(2020), MF-He-Sun-Zhang JHEP01(2024)079, Chen-Chen-MF-He, work in progress

Motivations of this paper

- QFT with negative energy
  - The IR behavior of $D=4$ $N=4$ SYM on S^1 is $d=3$ pure YM theory
  - The negative Casimir energy of a gauge theory
    ~ The mass of an AdS soliton

Witten `98, Horowitz-Myers `98

- DOF from EE (e.g. the coefficient of the A-type anomaly for a spherical entangling surface in CFT, *Solodkhin `08*)
  - Renormalized EE with a spherical entangling *Liu-Mezei `12*
  - The entropic C function from the EE with the striped subsystem

Nishioka-Takayanagi `06

Motivations of this paper

- What is difference between two entropy? → trace anomaly part
  - Renormalized EE only contains either A-type or B-type anomaly
  - Entropic c function contains both A-type and B-type anomalies

The entanglement entropy (an extension of the thermal entropy)

- System whose total Hilbert space is a direct product:

$$H = H_A \otimes H_B$$

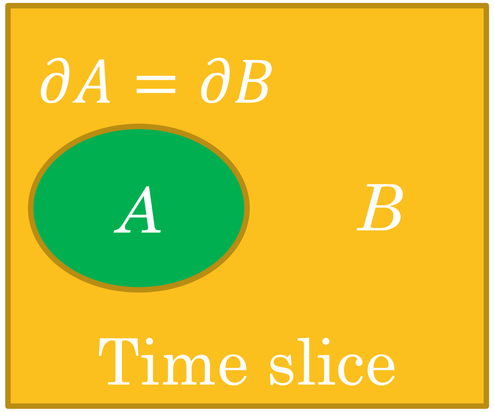
- Definition of the reduced density matrix $\rho_A = Tr_B(\rho)$ taking the trace over H_B

- Entanglement Entropy (EE) defined using the density matrix ρ_A as

$$S_A = -Tr_A(\rho_A log \rho_A)$$

Von Neumann entropy of ρ_A

- In QFT, A and B: often a spatial bipartition of a system



The UV structure of entanglement entropy

- S_{UV} with the entangling surface $S^1 \times S^{d-3}$ vs $S_{UV,0}$ with the surface S^{d-2}
- The UV structure of entanglement entropy is $S_{UV} = \frac{L_\phi}{R} S_{UV,0}$

$$S^A \sim \gamma \frac{Area(\partial A)}{a^{d-1}}$$

$+subleading\ terms$

$$S_{ren} = \frac{1}{R} f_d(R\partial_R) R S_{EE} = L_d(R\partial_R) S_{EE}$$

$$= \begin{cases} \frac{1}{(d-2)!!} R\partial_R (R\partial_R - 2) \ldots (R\partial_R - (d-3)) S_{EE}, & d = \text{odd}, \\ \frac{1}{(d-2)!!} (R\partial_R + 1)(R\partial_R - 1) \ldots (R\partial_R - (d-3)) S_{EE}, & d = \text{even}, \end{cases}$$

$$f_d(R\partial_R) S_{EE}^{(0)} = \begin{cases} \frac{1}{(d-2)!!} (R\partial_R - 1)(R\partial_R - 3) \ldots (R\partial_R - (d-2)) S_{EE}^{(0)}, & d = \text{odd}, \\ \frac{1}{(d-2)!!} R\partial_R (R\partial_R - 2) \ldots (R\partial_R - (d-2)) S_{EE}^{(0)}, & d = \text{even}. \end{cases}$$

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$$f_d(R\partial_R) S^{(0)}_{EE} = \begin{cases} \frac{1}{(d-2)!!} (R\partial_R - 1)(R\partial_R - 3)\ldots(R\partial_R - (d-2)) S^{(0)}_{EE}, & d = \text{odd}, \\ \frac{1}{(d-2)!!} R\partial_R (R\partial_R - 2)\ldots(R\partial_R - (d-2)) S^{(0)}_{EE}, & d = \text{even}. \end{cases}$$

Equivalent to D-1 dimensional renormalized entanglement entropy on $R^{1,d-2}$

Odd dimensional results

$$S_{ren} = R\partial_R S_{EE} \quad \text{for } d = 3,$$
$$S_{ren} = \frac{1}{3}R\partial_R(R\partial_R - 2)S_{EE} \quad \text{for } d = 5.$$

- Through Kaluza-Klein reduction along S^1, the renormalized entanglement entropy effectively embodies d-1 dimensional one in the low-energy limit
- In systems respecting Lorentz symmetry, the 2-dimensional entropic c-function $C = R\,{}^{\partial S}/_{\partial R}$ is both non-negative and monotonically decreasing
- The behavior of the 4d renormalized entanglement entropy displays non-monotonic tendencies

Renormalized entanglement entropy of 4d QFT (even dimensional results)

- The length scale R_1 of the subregion related to rescaling of the metric

$$R_1\partial_{R_1}S_A = \lim_{n\to 1}(-2\partial_n \int d^{d+1}x g_{\mu\nu}(x)\frac{\delta}{\delta g_{\mu\nu}(x)}[w - nw|_{n=1}])$$
$$\frac{1}{2\pi}\lim_{n\to 1}\partial_n\left(\langle\int d^{d+1}x\sqrt{g}T_\mu{}^\mu(x)\rangle_{M_n} - n\langle\int d^{d+1}x\sqrt{g}T_\mu{}^\mu(x)\rangle_{M_1}\right)$$

- The log term of the entanglement entropy $S_{EE} = s\log\left(\frac{\epsilon}{R_1}\right) + ..$

$$s = \frac{a}{180}\int_{\partial\sigma} d^2x\sqrt{h}E_2 + \frac{c}{120}\int_{\partial\sigma} d^2x\sqrt{h}I_2$$

- For cylinder type topology: $S_{ren} = \frac{1}{2}(l\partial_l + 1)(l\partial_l - 1)S_{EE} = s = \frac{cL_\phi}{240l}.$ Only B-Type anomaly
- For a spherical surface S^2: $S_{ren} = \frac{1}{2}R\partial_R(R\partial_R - 2)S_{EE} = s = \frac{a}{90}$ Only A-Type anomaly

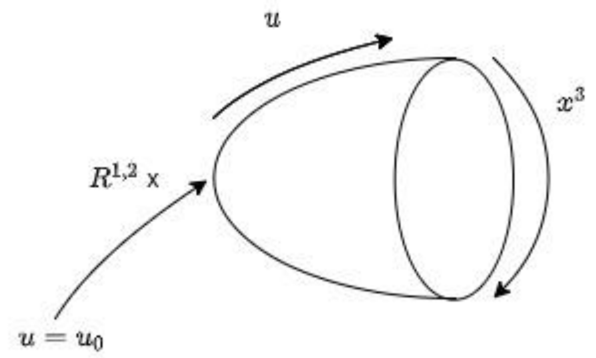
The AdS soliton dual to confining theory

- The double Wick rotation of the AdS black hole

- It corresponds to the ground state of QFT with the anti-periodic boundary condition on fermions

$$ds^2 = \frac{L^2}{u^2}\left(-dt^2 + \frac{du^2}{f(u)} + f(u)d\varphi^2 + \sum dx^i dx^i\right),$$

where $f(u) = 1 - \left(\frac{u}{u_0}\right)^4$



- The mass of the AdS soliton = negative energy

The AdS soliton with a background gauge field

- The metric of the AdS soliton with a background gauge field

$$ds^2_{d+1} = \frac{L^2}{z^2}\left(\frac{dz^2}{f_d(z)} + f_d(z)d\phi^2 - dt^2 + dR^2 + R^2 d\Omega_{d-3}\right) \qquad f_d(z) = 1 - \left(1 + \frac{\epsilon_1 z_+^2 a_\phi^2}{\gamma^2}\right)\left(\frac{z}{z_+}\right)^d + \frac{\epsilon_1 z_+^2 a_\phi^2}{\gamma^2}\left(\frac{z}{z_+}\right)^{2(d-1)},$$

a_ϕ a constant gauge field and $\gamma^2 = \frac{(d-1)g_e^2 L^2}{(d-2)\kappa^2}$

$$A_\phi = a_\phi\left(1 - \left(\frac{z}{z_+}\right)^{d-2}\right),$$

- The Kaluza-Klein mass

$$M_0 = \frac{1}{4\pi z_+}\left(d - \frac{\epsilon_1 (d-2) z_+^2 a_\phi^2}{\gamma^2}\right) > 0$$

- The solution exists if $a_\phi \leq \frac{2\pi M_0 \gamma}{\sqrt{d(d-2)}} = a_c$

The boundary energy

- $M = \langle T_{00} \rangle V_{d-2}/M_0$

$$M = -\frac{V_{d-2}}{M_0}\frac{L^{d-1}\bar{a}_\phi}{2\kappa^2 z_+^d} \qquad \bar{a}_\phi = 1 - \left(z_+ a_\phi\right)^2$$

- The boundary energy changes the sign when we change Wilson lines a_φ

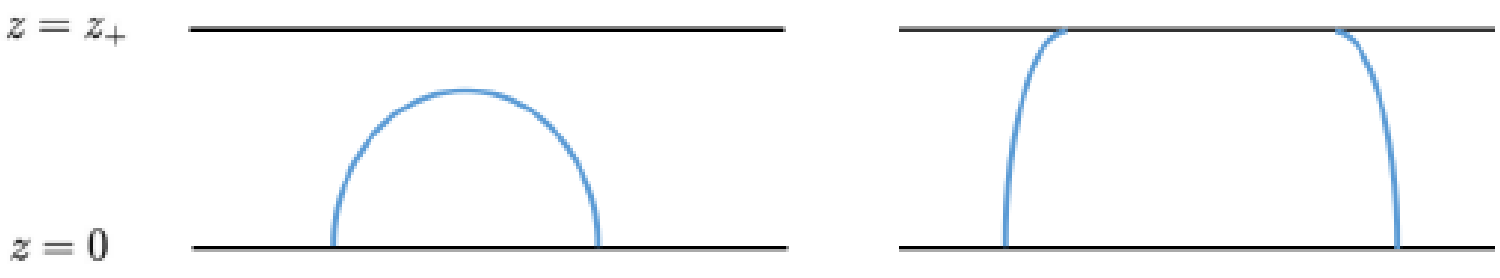
$$\begin{cases} M < 0 & z_+ a_\phi < 1 \\ M > 0 & z_+ a_\phi > 1. \end{cases}$$

For $a_\varphi = 0$, it realizes Casimir energy of 4d SYM theory.
Casimir energy is different among periodic and anti-periodic b.c.

The holographic entanglement entropy

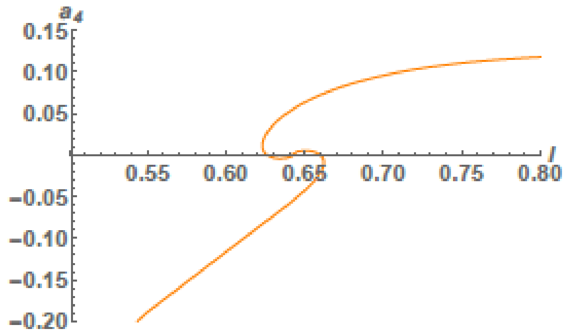
- The boundary region $S^1 \times S^{d-3}$: $\mathrm{R} = l$ and $0 \leq \phi \leq L_\phi$
- The surface action: $$A = \int d^{d-1}x\mathcal{L} = \Omega_{d-3} L_\phi L^{d-1} \int dz \frac{R^{d-3}}{z^{d-1}} \sqrt{1 + f\dot{R}^2}$$
- Boundary conditions:

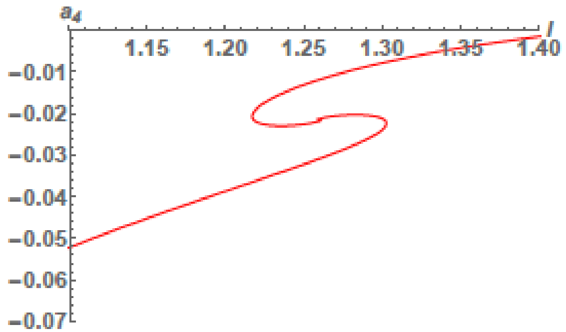
Disk type: $R(z_t) = 0, R'(z_t) = \infty$ A cylinder: $z_t = z_+$



The coefficient a_4 in $d=4$

- The AdS boundary expansion: $$R(z) = l - \frac{z^2}{4l} + a_4(l)z^4 + \frac{z^4}{32l^3}\log\frac{z}{l} + \dots$$
- $a_4(l)$ must be determined by the IR boundary condition



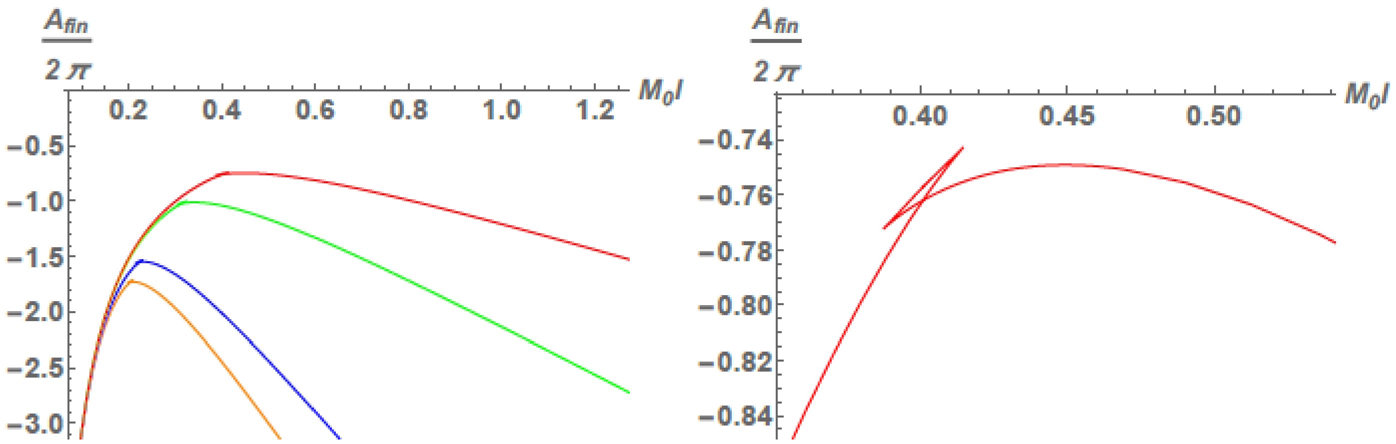


Left: $a_\varphi = i/2$ Right: $a_\varphi = 1/\sqrt{2}$

For small l, disk type surface is favored
For large l, cylinder type surface is favored

HEE in *4d*

- Finite part of HEE shows HEE with the Wilson line increases (vs confinement)



- Left: from $a_\phi = \frac{i}{2}, 0, \frac{2}{3}, 1/\sqrt{2}$ from the left to the right
- Swallow tail phase transition

An interpretation of HREE

- The renormalized EE can capture effective DOF on S^1:
  - EE can detect effective DOF if $E \sim 1/l > M_0$
  - EE can not detect effective DOF if $E \sim 1/l < M_0$

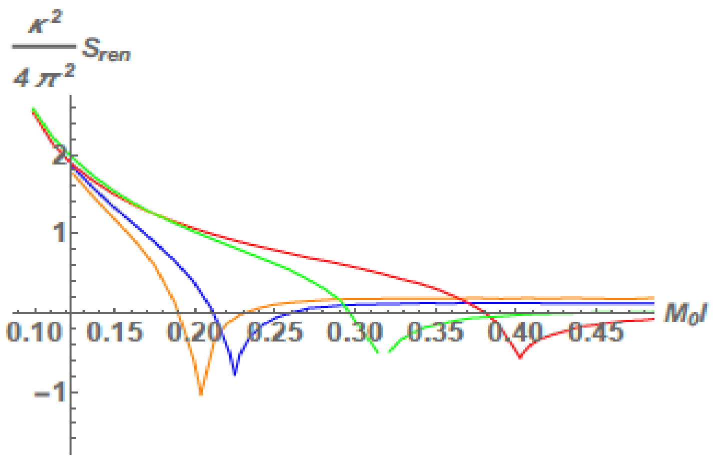
$$\frac{\kappa^2 S_{ren}}{4\pi^2 L_\phi} = -4l^2 a_4(l) - 2l^3 a_4'(l) + \frac{7}{64l} - \frac{A_{fin}}{2}$$

$$= -4l^2 a_4(l) - 2l^3 a_4'(l) + 2\int^l l' a_4(l') dl' + \frac{1}{8l} + c_r,$$

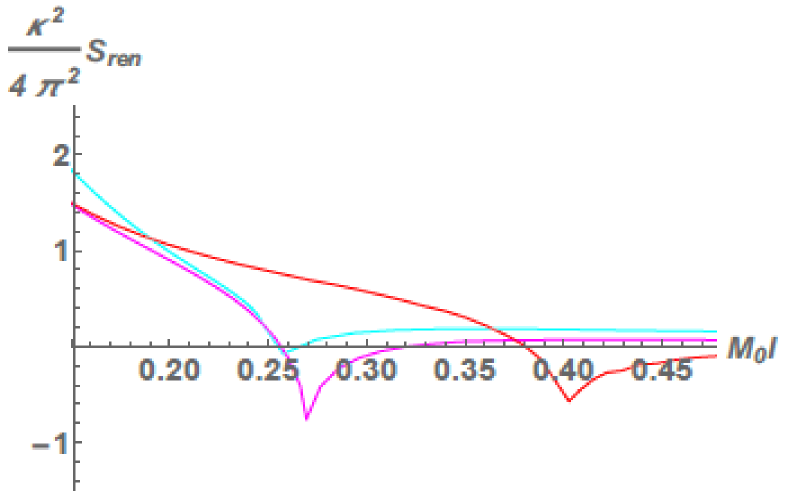
- Breaking conformal invariance
- Terms brought from the Weyl anomaly (B-type anomaly)

Renormalized EE in *4d*

- REE corresponds to effective DOF from EE with spherical surface
- REE non-monotonically behaves near critical lengths



$a_\phi = \frac{i}{2},\ 0,\ \frac{2}{3},\ \frac{1}{\sqrt{2}}$ from the left to the right

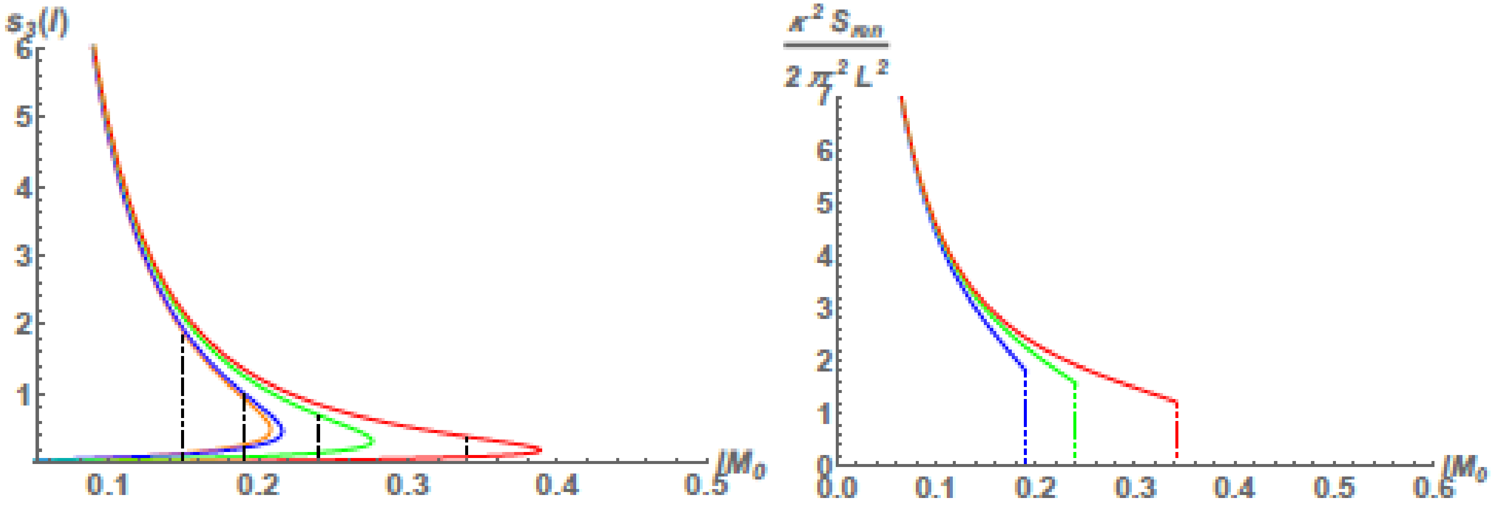


$M_0 = 1/\pi,\ 2/5,\ 3/5$ from the right to the left

- It implies that Wilson lines make particles light
- Massive modes decouple others soon

Renormalized EE in *3d*

- 3d: REE is positive and monotonically decreasing.
  - consistent with the entropic c function $C = {}^{l dS}/_{dl}$ in $R^{1,1}$



$a_\phi = i/2,\ 0,\ 1,\ 2/\sqrt{3}$ from the left to the right

  - $s_3 \sim {}^{dS}/_{dl}$. The minimal surface is concave: $S'' < 0$

Spectrum of spin 0^{++} glueball-like operators for *d=4*

- Decrease with increase of energy *M* (also in other dimensions)

- $M_0 = \frac{1}{\pi}$

$$m^2 = 11.6,\quad 34.5,\quad 69.0,\quad 115,\ldots \qquad \text{for } a_\phi = 0,$$
$$m^2 = 3.60,\quad 11.1,\quad 22.4,\quad 37.3,\ldots \qquad \text{for } a_\phi = \frac{1}{\sqrt{2}},$$
$$m^2 = 19.8,\quad 58,\quad 115,\quad 192,\ldots \qquad \text{for } a_\phi = i.$$

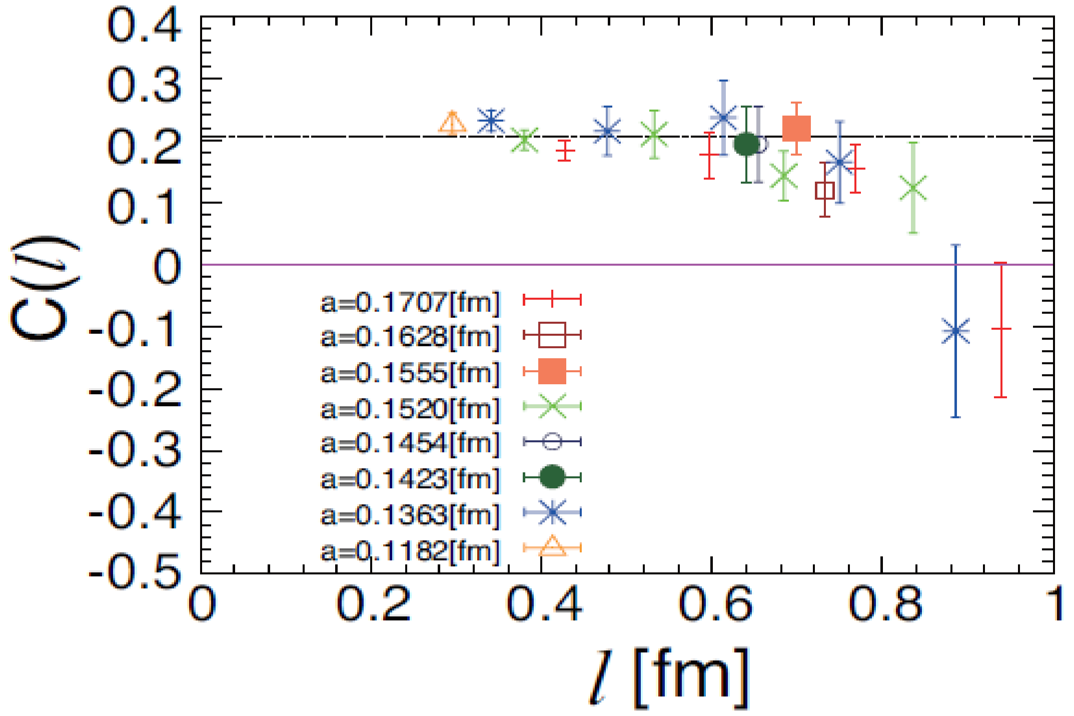
- $M_0 = 0$
  (the extremal case)

$$m^2 = 0.52,\quad 0.94,\quad 1.60,\quad 2.49,\ldots \qquad \text{for } a_\phi = i/2,$$
$$m^2 = 1.94,\quad 3.21,\quad 5.32,\quad 8.27,\ldots \qquad \text{for } a_\phi = i,$$
$$m^2 = 4.75,\quad 9.80,\quad 18.2,\quad 30.9,\ldots \qquad \text{for } a_\phi = 2i,$$

Discussion

Renormalized EE	Generalized entropic C function $C(l) = \frac{l^{d-1}}{V}\frac{dS}{dl}$
Increase at a region However, $S_{ren}(l \to 0) \geq S_{ren}(l \to \infty)$ (Decrease and positive for d=3)	Increase at a region and positive However, $S_{ren}(l \to 0) \geq S_{ren}(l \to \infty)$
Counting the effective DOF of Wilson lines along the S^1 direction	Counting the effective DOF of Wilson lines along the S^1 direction
Quantum phase transition	Quantum phase transition

Generalized entropic C-function of SU(3) Yang-Mills theory on the lattice *Itou-Nagata-Nakagawa-Nakamura-Zakharov '15*



Entropic C function

The black line: C=2.06

Decrease in the middle l=0.88 fm

Agreement with the critical temperature $T_c^{-1} = 0.714\text{fm}$ ($T_c = 280$ MeV)
and the Lambda scale Λ_{MS}^{-1} ~0.8 fm

The critical length of our holographic model: $l \sim M_0^{-1}$ $(M_0 \sim T_c, \Lambda_{MS})$

However, our holographic model is relevant to large N YM only in the IR region.

Dual theory consists of many matter fields

Thank you!