# Quantum Solitons

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# Motivation: Confinement

Strong interactions are described by QCD, a quantum field theory of quarks $q$ and gluons $g$

Time evolution in QCD is generated by a Hamiltonian $H(q, g)$ which is known

The spectrum of stationary configurations consists of the eigenstates of $H$

They consist of stable nuclei

The spectrum does not obviously include the quarks or the gluons

This situations is called *confinement*

# Paradigms with Solitons

No one knows why confinement occurs

Two of the leading proposals are:

1) Dual superconductivity: A condensate of color monopoles causes color flux to collimate into fluxtubes

2) The vacuum is permeated by a network of vortices

Both the fluxtubes and the vortices have been observed in lattice simulations

The big questions are *why they arise*, and first *how to describe them*

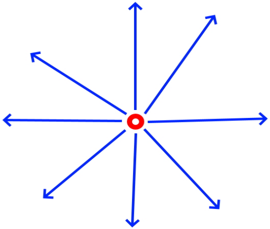
Monopoles and vortices are examples are solitons

Classically, solitons (in most models) increase the energy, and so do not appear in the lowest energy configuration

$\Rightarrow$ Confinement can only be caused by *quantum* solitons

# Dual Superconductivity 1

In QCD and Yang-Mills, colored sources source color flux $F$



If $F$ were isotropic, then $F \sim 1/r^2$

And so the energy on each shell would be

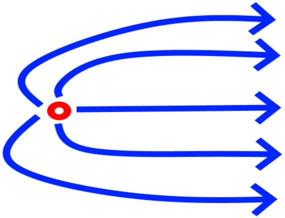
$$E \sim \int_{S^2} F^2 \sim 1/r^2$$

whose integral converges at large $r$ $\Rightarrow$ Finite energy

So, like QED, there would be no confinement

# Dual Superconductivity 2

If, on the other hand, the flux was collimated in a tube



The energy in each shell would be the tube tension, which is constant at large $r$

$\Rightarrow$An infinite energy is required for a colored source

This would result in confinement

# Dual Superconductivity 3: The Meissner Effect

Why should the color flux all be collimated in a tube?

In a superconductor, the magnetic flux is collimated into tubes, called Abrikosov vortices

In that case, this is a result of the Meissner Effect

Meissner effect: The vacuum contains an electric condensate which makes the photon massive, and so the flux is exponentially suppressed in the vacuum

The flux needs to escape by Gauss' Law, and so it creates a tube which is not in the vacuum state and passes through this tube

Dual Meissner hypothesis ('t Hooft and Mandelstam):
The same mechanism is at work in QCD, a magnetic monopole condensate in the vacuum repels the color flux

$\Rightarrow$ Need to understand the magnetic monopoles

# Solitons

Classically, magnetic monopoles are solitons

What are solitons?

Solitons are solutions of the classical equations of motion defined by the following properties

1) They are localized, at least in some directions

2) They never decay

3) They are intrinsically nonlinear (no solution in the linear limit)
The solution is proportional to inverse powers of the coupling

# Example: Solitons in 1+1d Scalar Models

Consider a classical scalar field theory described by the Hamiltonian density

$$\mathcal{H}(x) = \frac{\pi^2(x)}{2} + \frac{(\partial_x \phi(x))^2}{2} + \frac{V(\sqrt{\lambda}\phi(x))}{\lambda}$$

We are interested in a perturbative expansion in $\lambda$

There will be solutions that exist at small $\phi(x)$, which essentially solve the linearized equations of motion

Consider intrinsically nonlinear solutions $\phi(x,t) = f(x,t)$, which require the cubic or higher terms in the potential $V$

As $\mathcal{H}$ may be written as a function of $\sqrt{\lambda}\phi(x)$, one expects that these are of order

$$f(x,t) \sim O(1/\sqrt{\lambda})$$

In other words, intrinsically nonlinear solutions, such as solitons, are nonperturbative

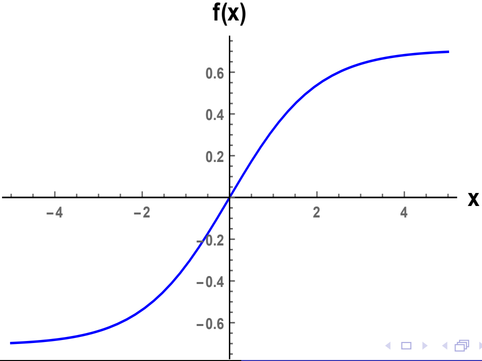
# Example: The $\phi^4$ Kink

Consider the double well model

$$V(\sqrt{\lambda}\phi) = \frac{\left(\lambda\phi^2 - \frac{m^2}{2}\right)^2}{4}$$

There will be a kink solution

$$\phi(x,t) = f(x) = \frac{m}{\sqrt{2\lambda}}\tanh\left(\frac{mx}{2}\right)$$



# The Fock Space

**What is the quantum state corresponding to the classical solution $\phi(x,t) = f(x)$?**

Perturbation theory gives us states in the Fock space (We will call it the *vacuum sector*)

This is constructed in the Schrodinger picture by decomposing the field

$$\phi(x) = \int \frac{dp}{2\pi} \frac{a_p^\dagger + a_{-p}}{\sqrt{2\omega_p}} e^{-ipx}$$

and its conjugate $\pi(x)$ and acting on the vacuum $|\Omega\rangle$ ($a_p|\Omega\rangle = 0$) with finite numbers of $a^\dagger$

$$|p_1 \cdots p_n\rangle = a_{p_1}^\dagger \cdots a_{p_n}^\dagger |\Omega\rangle$$

# No Solitons in the Fock Space

Perturbation theory gives us states in the vacuum sector Fock space

$$|\psi\rangle = \sum c_{p_1 \cdots p_n} |p_1 \cdots p_n\rangle$$

with generic coefficients $c$

Is one of these the kink?

$$\frac{\langle\psi|\phi(x)|\psi\rangle}{\langle\psi|\psi\rangle} \sim O(\lambda^0)$$

**NO.** The kink has $\phi$ of order $O(1/\sqrt{\lambda})$

We learn that perturbative states correpond to values of $\phi$ near zero

Indeed, perturbation theory is a moment expansion with small moments of $\phi$

# The Displacement Operator

Somehow we need to shift

$$\phi \to \phi + f$$

The key to our construction is the *displacement operator*

$$\mathcal{D}_f = \mathrm{Exp}\left[-i \int dx \pi(x) f(x)\right]$$

It does exactly what we want

$$\phi \mathcal{D}_f = \mathcal{D}_f (\phi + f)$$

It can do this in any functional

$$\mathcal{D}_f^\dagger F[\phi, \pi] \mathcal{D}_f = F[\phi + f, \pi]$$

and it commutes with normal ordering

Now consider the state $\mathcal{D}_f|\Omega\rangle$
This is called a *coherent state* and it is not in the Fock space

The expectation value is

$$\langle\Omega|\mathcal{D}_f^\dagger\phi(x)\mathcal{D}_f|\Omega\rangle = \langle\Omega|(\phi(x)+f(x))|\Omega\rangle = f(x)$$

It is close to our kink solution $f(x)$!

Is it our kink state?

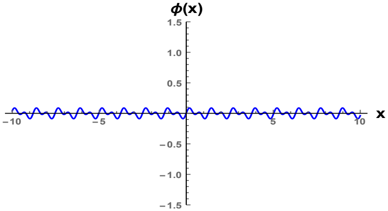
Define the *kink sector* to be the Fock space made by acting a finite number of creation operators on the kink ground state

The coherent state $\mathcal{D}_f|\Omega\rangle$ is in the kink sector, as is $\mathcal{D}_f|\psi\rangle$

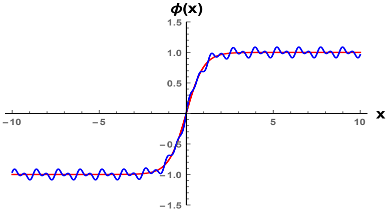
Conversely, any kink sector states can be created by acting with $\mathcal{D}_f$ on a vacuum sector state $|\psi\rangle$

# Summary So Far

We started with the Fock space aka vacuum sector states $|\psi\rangle$, which look like




Then we acted on them with the displacement operator to get the coherent states $\mathcal{D}_f|\psi\rangle$



# What next?

We have constructed a state in the kink sector

The same construction works for any soliton, with a tweak if the soliton is time-dependent (See Kehinde's poster)

But we want to do more:

We want to:

1) Find the spectrum of solitons and their excitations
2) Calculate scattering amplitudes involving solitons
3) Calculate the decay rates of unstable soliton excitations

We need a good formalism for treating these soliton states

# List of Sacrifices

Every formalism for treating quantum solitons requires a sacrifice.

The old formalisms require sacrifices (1)-(4)

1) Only works at one loop (spectral methods, classical-quantum correspondence)
2) Only works for theories with SUSY or integrability
3) Numerics becomes unwieldly beyond 1+1 dimensions (Hamiltonian truncation)
4) Calculations are very complicated/impractical (collective coordinates)
5) Global symmetries are not manifest (Linearized Soliton Perturbation Theory)

Our group has constructed Linearized Soliton Perturbation Theory, which instead sacrifices (5)

# The Basic Problems

There are two basic problems in physics

1) Find the spectrum

2) The initial value problem

In quantum physics, these two problems can be restated:

1) Find the eigenstates of $H$

2) Act on states with $e^{-iHt}$

Let's start with the first problem.

We need to solve

$$H\mathcal{D}_f|\psi\rangle = E\mathcal{D}_f|\psi\rangle$$

However the state $\mathcal{D}_f|\psi\rangle$ is not in our Fock space
It has moments that are inverse powers of the coupling
This problem is hopelessly nonperturbative

# The Soliton Hamiltonian

We are saved by a miracle, or a sleight-of-hand

Let us rewrite our eigenvalue problem

$$H\mathcal{D}_f|\psi\rangle = E\mathcal{D}_f|\psi\rangle$$

as

$$\mathcal{D}_f\left(\mathcal{D}_f^\dagger H\mathcal{D}_f\right)|\psi\rangle = \mathcal{D}_f E|\psi\rangle$$

Let us called the combination

$$H' = \mathcal{D}_f^\dagger H\mathcal{D}_f$$

the *soliton Hamiltonian*

Now our problem is

$$H'|\psi\rangle = E|\psi\rangle$$

Miracle: It now only involves the state $|\psi\rangle$, which is in the perturbative Fock space

# Summary: The Soliton Hamiltonian

Summarizing, we have replaced the horrible nonperturbative spectral problem

$$H\mathcal{D}_f|\psi\rangle = E\mathcal{D}_f|\psi\rangle$$

with the eigenvalue problem

$$H'|\psi\rangle = E|\psi\rangle$$

which we can solve perturbatively, as $|\psi\rangle$ is in the perturbative Fock space

Good for the spectrum. What about dynamics?
Time evolution is given by

$$|t\rangle = e^{-iHt}\mathcal{D}_f|\psi\rangle$$

which is horrible and nonperturbative but equivalent to

$$|t\rangle = \mathcal{D}_f e^{-iH't}|\psi\rangle$$

in which we need only act $H'$ on our perturbative state $|\psi\rangle$

# Did we really solve the problem?

We need to act with $\mathcal{D}_f$ at the end, but this is unitary and so does not affect amplitudes/matrix elements/*etc.* and so in general we can forget about it

Is the operator $H'$ really perturbative?

Well, we can calculate it and check.
Just use our old formula:

$$H'[\phi,\pi] = \mathcal{D}_f^\dagger H[\phi,\pi]\mathcal{D}_f = H[\phi+f,\pi]$$

We can expand this in powers of the coupling to see whether it is actually perturbative

$$H' = \sum_i H'_i, \quad H'_i \sim O(\lambda^{i/2-1})$$

The components $H'_0$, $H'_1$ and $H'_2$ are not suppressed by any powers of the coupling $\lambda$

# Summary: The Soliton Hamiltonian

We have recast the spectral problem and time evolution problem in terms of the soliton Hamiltonian $H'$

Now the problem looks perturbative because the states are perturbative

But we need to be careful because $H'$ contains three components $H'_0$, $H'_1$ and $H'_2$ which are not suppressed by powers of the coupling

We can only solve our problems perturbatively *if* we can first simultaneously diagonalize these three operators exactly

Can we do it?

For concreteness, let us restrict our attention to $(1+1)$-dimensional theories with a single scalar field

In higher dimensions we will have UV divergences: In a preprint posted to the arXiv last week we have shown that a consistent treatment with counterterms allows the calculation below to be extended up to 3+1 dimensions

In $(1+1)$-dimensions we will be interested in kinks, although much of the discussion below applies almost as-is to oscillons

Our preprint extended this treatment to domain walls, although a similar treatment will exist for Skyrmions, lumps, *etc.* ...

# Expanding in the Coupling $\lambda$

Let $Q$ be the mass of the ground state kink

This is the difference in energy between the minimial energy configuration with and without a kink

We will expand it in powers of the coupling

$$Q = \sum_{i=0}^{\infty} Q_i$$

$Q_i$ is of order $O(\lambda^{i-1})$.

Recall that we also expand the kink Hamiltonian

$$H' = \sum_{i=0}^{\infty} H'_i$$

$H'_i$ has $i$ powers of the fields, and is of order $O(\lambda^{i/2-1})$.

# The First Few Orders

$H'_0$ is of order $O(1/\lambda)$, $H'_1 \sim O(1/\sqrt{\lambda})$, and $H'_2 \sim O(\lambda^0)$

Therefore we can only begin our perturbative expansion if these three operators can be diagonalized exactly

Recalling that $H' = \mathcal{D}_f H \mathcal{D}_f^\dagger$ we may calculate them

$$H'_0 = Q_0$$

This is just the classical kink mass. It is a scalar and so anything diagonalizes it

Next is the tadpole term: $H'_1$ vanishes as $f(x)$ is a solution of the classical equations of motion

$$H'_1 = 0$$

So if we can diagonalize $H'_2$, a perturbative expansion is possible

# The Free Kink Hamiltonian

To remove divergences, we need to normal order

Now the free part of the kink Hamiltonian is

$$H_2' = \frac{1}{2}\int dx \Big[: \pi^2(x) :_a + : (\partial_x \phi(x))^2 :_a + V''[\sqrt{\lambda} f(x)] : \phi^2(x) :_a\Big]$$

It looks like a free massive scalar, but the mass is position-dependent

If the mass were position-independent, we could diagonalize it using the usual $a$ and $a^\dagger$ operators resulting from a decomposition of $\phi(x)$ and $\pi(x)$ into plane waves

But it's not so, how do we diagonalize this?

# Normal Modes

In the vacuum sector, the free Hamiltonian $\int dp\omega_p a_p^\dagger a_p$ is diagonalized by the operators $a^\dagger$ and $a$ which create plane waves

This is reasonable, as plane waves are the solutions of the linearized equations of motion

In the kink sector, the linearized equation of motion is the Sturm Liouville equation

$$V^{(2)}[\sqrt{\lambda}f(x)]\mathfrak{g}(x) = \omega^2\mathfrak{g}(x)+\mathfrak{g}''(x), \qquad \phi(x,t) = f(x)+e^{-i\omega t}\mathfrak{g}(x).$$

The solutions $\mathfrak{g}(x)$ are normal modes

The normal modes in the soliton sector will play the roles played by the plane waves in the vacuum sector

# Classification of Normal Modes

Normal modes are classified by their frequencies: $\omega$

1) $(\omega = 0)$ There is one zero mode

$$\mathfrak{g}_B(x) = -\frac{f'(x)}{\sqrt{Q_0}}$$

2) $(\omega \geq m)$ For each real number $k$ there is a continuum normal mode $\mathfrak{g}_k(x)$ with

$$\omega_k = \sqrt{k^2 + m^2}$$

3) $(m > \omega_S > 0)$ Sometimes there are discrete shape modes $\mathfrak{g}_S(x)$

# Normal Mode Decomposition

In the Schrodinger picture, we are free to decompose $\phi(x)$ and $\pi(x)$ in any basis

As the normal modes satisfy a Sturm-Liouville equation, they provide a basis of the space of functions, and so we may use them to decompose the fields

$$\phi(x) = \phi_0 \mathfrak{g}_B(x) + \sum\!\!\!\!\!\!\int \frac{dk}{2\pi} \frac{\left(b_k^\dagger + b_{-k}\right)}{\sqrt{2\omega_k}} \mathfrak{g}_k(x)$$

$$\pi(x) = \pi_0 \mathfrak{g}_B(x) + i \sum\!\!\!\!\!\!\int \frac{dk}{2\pi} \sqrt{\frac{\omega_k}{2}} \left(b_k^\dagger - b_{-k}\right) \mathfrak{g}_k(x)$$

where we have defined the shorthand

$$b_{-S} = b_S, \quad \sum\!\!\!\!\!\!\int \frac{dk}{2\pi} = \int \frac{dk}{2\pi} + \sum_S$$

# Summary: Operators

Let us explain this another way

Any operator in the theory can be written in terms of the field $\phi(x)$ and its dual $\pi(x)$

But these are linearally related to another basis $a_p^\dagger$ and $a_p$

So we can choose whether to write any operator in terms of $\{\phi(x), \pi(x)\}$ or $\{a_p^\dagger, a_p\}$.

But now we have introduced a third basis $\{b_k^\dagger, b_k, b_S^\dagger, b_S, \phi_0, \pi_0\}$

The great thing about the Schrodinger picture is that we can choose to write any operator in any of the three bases, using whichever makes life simpler

# Diagonalizing the Free Kink Hamiltonian

What have we gained?

Written in this basis, the free kink Hamiltonian is

$$H'_2 = Q_1 + \frac{\pi_0^2}{2} + \sum\!\!\!\!\!\!\int \frac{dk}{2\pi}\omega_k b_k^\dagger b_k$$

It is a sum of solved quantum mechical systems: the free particle and the QHO

So we know how to diagonalize it. The ground state $\mathcal{D}_f|0\rangle_0$ is

$$\pi_0|0\rangle_0 = b_k|0\rangle_0 = b_S|0\rangle_0 = 0.$$

As $\{b^\dagger, b, \pi_0, \phi_0\}$ are related by the usual creation and annihilation operators $\{a^\dagger, a\}$ by a linear Bogoliubov transform:
$|0\rangle_0$ is a squeezed state and $\mathcal{D}_f|0\rangle_0$ is a squeezed coherent state

We can excite mesons (perturbative excitations) using $b_k^\dagger$ and also internal shape modes using $b_S^\dagger$

# Summary: The Spectra of Soliton States at One-Loop

So, we have diagonalized the Hamiltonian up to corrections of order $O(\sqrt{\lambda})$

This means that we have already diagonalized $H'$ at leading order

So we know the soliton excitation spectrum: One soliton plus its excitations and finite numbers of perturbative quanta (mesons)

The eigenvalues that we have calculated $Q_0 + Q_1$ plus the $\omega_k$ that comes from $b^\dagger$ operators are the tree level and one-loop mass corrections

The higher corrections to $H'$ are suppressed by powers of $\sqrt{\lambda}$ and so can be computed in perturbation theory

# Interactions

The interaction terms are quite simple

$$H'_{n>2} = \lambda^{\frac{n}{2}-1} \int dx \frac{V^{(n)}(\sqrt{\lambda} f(x))}{n!} :\phi^n(x):_a$$

It is convenient to rewrite them in terms of $::_b$ normal ordering, which places the $b^\dagger$ and $\phi_0$ first, because $b$ and $\pi_0$ annihilate $|0\rangle_0$

This can be done with Wick's theorem

$$:\phi^j(x):_a = \sum_{m=0}^{\lfloor \frac{j}{2} \rfloor} \frac{j!}{2^m m!(j-2m)!} \mathcal{I}^m(x) :\phi^{j-2m}(x):_b$$

$$\mathcal{I}(x) = \int \frac{dk}{2\pi} \frac{|\mathfrak{g}_k(x)|^2 - 1}{2\omega_k} + \sum_S \frac{|\mathfrak{g}_S(x)|^2}{2\omega_S}$$

We have used the $::_b$ normal-ordered interactions will be used to evolve states and so to compute scattering amplitudes and decay rates

# One Last Summary

Solitons in quantum field theory correspond to squeezed, coherent states $\mathcal{D}_f|\psi\rangle$ plus corrections

The corrections can be computed in perturbation theory, using the soliton Hamiltonian $H'$

Said differently, the states factorize into a perturbative part plus a nonperturbative squeeze and displacement operator
The nonperturbative parts can easily be written down using the classical solution

We have used this formalism to calculate kink-meson scattering amplitudes at the leading orders, the decay rates of excited kink states, form factors, etc.

This year our approach has been generalized to 2+1 and 3+1 dimensions as well as time-dependent solutions

# What next?

There is a lot of work to do, and we are always looking for collaborators, including:

1) Include fermions and gauge fields

2) Calculate instanton corrections

3) Quantize the oscillon, Nielsen-Olesen vortex and eventually the 't Hooft-Polyakov monopole

Our long term goal is to quantize the BPS monopole in $\mathcal{N}=2$ superQCD and try to follow the quantum state from weak coupling (large hypermultiplet mass) where it is described by the construction above, to strong coupling where it condenses and leads to confinement

If we can do it, we will have a first example of a monopole (state) in a theory with no classical monopole, and we can use it as an Ansatz for a QCD monopole, which can be tested on the lattice