Title: Consistent dynamical current operators in light-front quantum models

W.N.Polyzou
The University of Iowa

U.S. National Science Foundation

Outline

- **Light-front quantum mechanics**
- **Dynamics and currents**
- **Kinematic Weyl algebra**
- **Local gauge symmetry for non-local operators**
- **Dynamical currents - light front issues**

Light-front quantum mechanics

- **Hilbert space:** $\mathcal{H} = \oplus_n (\otimes_{i=1}^n \mathcal{H}_{m_i s_i})$
- **Kinematic light-front generators,** $\{P^+, \mathbf{P}_\perp, \hat{\mathbf{z}} \cdot \mathbf{J}, \mathbf{E}_\perp, \hat{\mathbf{z}} \cdot \mathbf{K}\}$ **are sums of single-particle generators,** $G_a = \sum G_{ai}$**.**
- **Dynamical generators** P^-**,** $\mathbf{J}_\perp$**:**

$$P^- := \frac{M^2 + \mathbf{P}_\perp^2}{P^+}$$

$$\mathbf{J}_\perp = \frac{1}{P^+}\left((P^+ - P^-)(\hat{\mathbf{z}} \times \mathbf{E}_\perp) - (\hat{\mathbf{z}} \times \mathbf{P}_\perp)\hat{\mathbf{z}} \cdot \mathbf{K} + \mathbf{P}_\perp \hat{\mathbf{z}} \cdot \mathbf{S} + M\mathbf{S}_\perp\right)$$

- **Light-front Bakamjian-Thomas construction:** $\mathbf{S} := \mathbf{S}_0$**.**
- **Dynamics:** $M^2 = M_0^2 + V$,

$$0 = [V, P^+] = [V, \mathbf{P}_\perp] = [V, \mathbf{E}_\perp] = [V, \hat{\mathbf{z}} \cdot \mathbf{K}] = [V, \mathbf{S}_0] = 0$$

- **Results in a dynamical unitary representation of the Poincaré group.**

Light-front quantum mechanics

- **Light-front Bakamjian-Thomas construction fails to satisfy cluster properties for transverse rotations ($N > 2$).**

$$\lim_{(x_i - x_j)^2 \to \infty} \| \left(U(\Lambda, a) - \otimes U_i(\Lambda, a) \right) T(x_1, \cdots, x_n) |\psi\rangle \| = 0$$

$$(\Lambda, a) = (R, 0)$$

- **Light-front Sokolov construction - inductive construction, starting from light-front $N = 2$-body Bakamjian-Thomas, construction, restores cluster properties, generates many-body interactions with light-front kinematic symmetry and dynamical $S_\perp$.**

Light-front quantum mechanics

- Sokolov construction results in 7 kinematic generators and 3 dynamical generators satisfying Poincaré commutation relations and cluster properties. The result of the construction is not unique.
- Connection with light front-field theory: Clothing operator method of Greenberg and Schweber, Shebeko, Shrikov, Kostylenko, Arslanaliev, uses a pseudo unitary operator ($W^*W = I$) to remove all terms from the interaction that have less than two annihilation operators.
- The transformed vacuum and one-particle states are exact eigenstates of the transformed light-front Hamiltonian in the clothed-particle representation.
- Transformed interactions are necessarily non-local, there are no vertices, and W is not unique. The transformed generators have same properties as generators in the Sokolov construction.

Light-front quantum mechanics

Phenomenology - practical considerations ($N = 2$)

$$M = \sqrt{m_1^2 + \mathbf{k}^2 + 2m_{red}V_{nr}} + \sqrt{m_2^2 + \mathbf{k}^2 + 2m_{red}V_{nr}}$$

$$\mathbf{k} := \mathbf{k}_r = \mathbf{k}_{nr}$$

- $M = M(H_{nr})$, **if** H_{nr} **reproduces measured (relativistically invariant** $S = e^{2i\delta(\mathbf{k})}$**) phase shifts as a function of** $\mathbf{k}_{nr}$**, then** M **reproduces the experimentally measured relativistically invariant phase shifts as a function of** $\mathbf{k}_r = \mathbf{k}_{nr}$**.**

- **Violations of cluster properties are small for** $N > 2$ **in the light-front BT construction.**

Dynamical constraints on current operators

$$[P^\mu, I_\mu(0)] = 0$$

$$U(\Lambda, a) I^\mu(x) U^\dagger(\Lambda, a) = (\Lambda^{-1})^\mu{}_\nu I^\nu(\Lambda x + a)$$

All components of the current operator can be expressed in terms of $I^+(0)$ and the dynamical generators

$$I^1(0) = -i[J^2, I^+(0)] \qquad I^2(0) = i[J^1, I^+(0)]$$

$$I^-(0) = I^+(0) - 2[J^1, [J^1, I^+(0)]]$$

Current conservation and current covariance give dynamical constraints on $I^+(0)$:

$$[P^+, I^-(0)] + [P^-, I^+(0)] - 2\sum_{i=1}^{2} [P^i, I^i(0) = 0$$

$$[J^2, [J^1, I^+(0)]] = 0 \qquad [J^1, [J^1, [J^1, I^+(0)]]] = [J^1, I^+(0)]$$

Dynamical constraints on current operators

- **Consistent covariant current operators necessarily have dynamical many-body parts.**
- **Commutator equations $\to$ dynamical parts are not unique.**
- **Current operators relate $\to$ current matrix elements with different initial and final states.**
- **The many-body parts of the current operator are representation dependent.**

$$T^\dagger T = I \qquad \lim_{x^+ \to \pm\infty} \| T e^{-iP_0^- x^+} |\psi\rangle \| = 0 \qquad \Leftrightarrow \qquad S' = S$$

$$U'(\Lambda, a) = TU(\Lambda, a)T^\dagger \qquad I'^\mu(0) = TI^\mu(0)T^\dagger$$

Dynamical constraints on current operators

Current operators?

- Given realistic relativistic light-front model of the dynamics - how do you get a current operator consistent with the dynamics?
- While current covariance and current conservation can be satisfied for matrix elements, in order to have predictive power for different initial and final states they need to be satisfied at the operator level.
- Proposal - construct a consistent conserved covariant current at the operator level using local gauge invariance in a non-local relativistic light-front model.

Dynamical constraints on current operators

Procedure

- Construct an irreducible set of operators out of kinematic light-front generators.
- Express dynamical Poincaré generators in the irreducible representation.
- Replace kinematic light-front translation generators by gauge covariant derivatives.
- Extract the operator coefficient of the term linear in the vector potential.

Light-front Weyl representation

$$X_i^- := \frac{i}{\sqrt{P_i^+}} K_i^3 \frac{1}{\sqrt{P_i^+}} = i \frac{\partial}{\partial P_i^+}_{s_f} \qquad X_i^j := \frac{i}{P_i+} E_i^j = i \frac{\partial}{\partial P_i^j}_{s_f} \qquad j \in \{1,$$

- **Commutation relations imply**

$$[X_i^-, P_j^+] = i\delta_{ij} \qquad [X_i^k, P_j^l] = i\delta_{ij}\delta_{kl}$$

all other commutators vanish.

- **These are kinematic operators. They are irreducible and can be used to construct any operator.**

Light-front "Weyl" representation

- **Irreducibility - a general operator O can be expressed as:**

$$O = \int \prod d\tilde{\mathbf{a}}_i d\tilde{\mathbf{b}}_i o(\tilde{\mathbf{a}}_1 \cdots \tilde{\mathbf{a}}_n, \tilde{\mathbf{b}}_1 \cdots \tilde{\mathbf{b}}_n) e^{i \sum_j \tilde{\mathbf{X}}_j \cdot \tilde{\mathbf{a}}_j} e^{i \sum_k \tilde{\mathbf{P}}_k \cdot \tilde{\mathbf{b}}_k}$$

$$\tilde{\mathbf{X}}_i = (X_i^1, X_i^2, X_i^-) \qquad \tilde{\mathbf{P}}_i = (P_i^1, P_i^2, P_i^+)$$

In these expressions $\tilde{\mathbf{X}}_i$ and $\tilde{\mathbf{P}}_k$ are operators, while the coefficients $\tilde{\mathbf{a}}_i = (a_i^+, \mathbf{a}_{i\perp})$ and $\tilde{\mathbf{b}}_i = (b_i^-, \mathbf{b}_{i\perp})$ are variables. $o(\tilde{\mathbf{a}}_1 \cdots \tilde{\mathbf{a}}_n, \tilde{\mathbf{b}}_1 \cdots \tilde{\mathbf{b}}_n)$ is an ordinary function of the variables that defines the operator.

Relation to matrix elements

$$o(\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2; \tilde{\mathbf{b}}_1, \tilde{\mathbf{b}}_2) =$$

$$(2\pi)^{-6} \int \prod_j d\tilde{\mathbf{p}}_j \langle \tilde{\mathbf{p}}_1 + \tilde{\mathbf{a}}_1, \tilde{\mathbf{p}}_2 - \tilde{\mathbf{a}}_2 | O | \tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2 \rangle e^{-i \sum \tilde{\mathbf{b}}_j \cdot \tilde{\mathbf{p}}_j}$$

$$\langle \tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2 | O | \tilde{\mathbf{p}}_1, \tilde{\mathbf{p}}_2 \rangle =$$

$$(2\pi)^{-3} \int d\tilde{\mathbf{a}} d\tilde{\mathbf{b}} o(\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2; \tilde{\mathbf{b}}_1, \tilde{\mathbf{b}}_2) e^{i \sum \tilde{\mathbf{a}}_i \cdot \tilde{\mathbf{x}}_i} e^{i \sum \tilde{\mathbf{b}}_j \cdot \tilde{\mathbf{p}}_j} e^{i \sum \tilde{\mathbf{x}}_k \cdot \tilde{\mathbf{p}}_k}$$

Gauge invariant non-local operators

Local gauge transformation on a light-front wave function

$$\psi(\tilde{\mathbf{x}}_1, \cdots, \tilde{\mathbf{x}}_n, x^+) \to e^{i\sum_j \chi(\tilde{\mathbf{x}}_j, x^+)}\psi(\tilde{\mathbf{x}}_1, \cdots, \tilde{\mathbf{x}}_n, x^+) \to$$

Requirement of local gauge invariance for non-local operators

$$e^{i\sum_j \chi(\tilde{\mathbf{x}}_j, x^+)}\langle \tilde{\mathbf{x}}_1 \cdots \tilde{\mathbf{x}}_n | O(x^+) | \tilde{\mathbf{y}}_1 \cdots \tilde{\mathbf{y}}_n \rangle =$$

$$\langle \tilde{\mathbf{x}}_1 \cdots \tilde{\mathbf{x}}_n | O(x^+) | \tilde{\mathbf{y}}_1 \cdots \tilde{\mathbf{y}}_n, \rangle e^{i\sum_j \chi(\tilde{\mathbf{y}}_j, x^+)}$$

Ensures matrix elements, $\langle \psi' | O(x^+) | \psi \rangle$, are gauge invariant.

Gauge invariant non-local operators

Use the parallel transport operator $W(x'^{+}, \tilde{\mathbf{x}}', \gamma, \tilde{\mathbf{x}}, x^{+})$

$$W(x'^{+}, \tilde{\mathbf{x}}', \gamma, x^{+}, \tilde{\mathbf{x}}) e^{i\chi(\tilde{\mathbf{x}}, x^{+})} = e^{i\chi(\tilde{\mathbf{x}}', x^{+\prime})} W(x'^{+}, \tilde{\mathbf{x}}', \gamma, x^{+\prime}; \tilde{\mathbf{x}}, x^{+}).$$

where γ **is a path between** $(\tilde{\mathbf{x}}, x^{+})$ **and** $(\tilde{\mathbf{x}}', x^{+\prime})$**. For** $\tilde{\mathbf{x}}' = \tilde{\mathbf{x}} + \delta\tilde{\mathbf{x}}$ **close to** $\tilde{\mathbf{x}}$ **and fixed** x^{+} **this becomes (short straight path)**

$$W(x^{+}, \tilde{\mathbf{x}} + \delta\tilde{\mathbf{x}}; x^{+}, \tilde{\mathbf{x}}) e^{i\chi(x^{+}\tilde{\mathbf{x}})} = e^{i\chi(x^{+}, \tilde{\mathbf{x}} + \delta\tilde{\mathbf{x}})} W(x^{+}, \tilde{\mathbf{x}} + \delta\tilde{\mathbf{x}}; x^{+}, \tilde{\mathbf{x}}).$$

Gauge invariant non-local operators

Covariant derivatives on the light front can be defined by taking the limit $\lambda \to 0$ of

$$\frac{f(x^+,\tilde{\mathbf{x}}+\lambda\delta\tilde{\mathbf{x}}) - \left(W(x^+,\tilde{\mathbf{x}}+\lambda\delta\tilde{\mathbf{x}},x^+;\tilde{\mathbf{x}}) - W(x^+,\tilde{\mathbf{x}};x^+,\tilde{\mathbf{x}}) + W(x^+,\tilde{\mathbf{x}};x^+,\tilde{\mathbf{x}})\right) f(x^+.\tilde{\mathbf{x}})}{\lambda}$$

The limit gives

$$\tilde{\mathbf{D}}f(x^+,\tilde{\mathbf{x}}) = \left(\tilde{\boldsymbol{\partial}} + \tilde{\boldsymbol{\partial}}_y W(x^+,\tilde{\mathbf{y}};x^+,\tilde{\mathbf{x}})_{\tilde{\mathbf{y}}=\tilde{\mathbf{x}}}\right) f(x^+,\tilde{\mathbf{x}})$$

This satisfies

$$e^{i\chi(x^+,\tilde{\mathbf{x}})}\tilde{\mathbf{D}}f(x^+,\tilde{\mathbf{x}}) = \tilde{\mathbf{D}}e^{i\chi(x^+,\tilde{\mathbf{x}})}f(x^+,\tilde{\mathbf{x}})$$

where

$$\tilde{\boldsymbol{\partial}}_y W(x^+,\tilde{\mathbf{y}};x^+,\tilde{\mathbf{x}})_{\tilde{\mathbf{y}}=\tilde{\mathbf{x}}} = -ie\tilde{\mathbf{A}}(x^+,\tilde{\mathbf{x}}).$$

Gauge invariant non-local operators

Replace the light-front momentum operators by light-front covariant derivatives in the Weyl representation:

$$\langle x^+, \tilde{\mathbf{x}}' | e^{i\tilde{\mathbf{p}}\cdot\tilde{\mathbf{b}}} | x^+, \tilde{\mathbf{x}}'' \rangle \to \langle x^+, \tilde{\mathbf{x}}' | e^{i(\tilde{\mathbf{p}} - e\tilde{\mathbf{A}}(x^+,\tilde{\mathbf{x}}))\cdot\tilde{\mathbf{b}}} | x^+, \tilde{\mathbf{x}}'' \rangle$$

Use the Trotter product formula to express functions of non-commuting $\tilde{\mathbf{p}}$ and $A(x^+, \tilde{\mathbf{x}})$ as products

$$(\textbf{above}) = \lim_{N\to\infty} \langle x^+, \tilde{\mathbf{x}}' | [e^{i\tilde{\mathbf{p}}\frac{\tilde{\mathbf{b}}}{N}} e^{-ieA(x^+,\tilde{\mathbf{x}})\cdot\frac{\tilde{\mathbf{b}}}{N}}]^N | x^+, \tilde{\mathbf{x}}'' \rangle$$

Limit of short straight paths

Dynamical currents

$$P^- = \int \prod d\tilde{\mathbf{a}}_i d\tilde{\mathbf{b}}_i p^-(\tilde{\mathbf{a}}_1 \cdots \tilde{\mathbf{a}}_n, \tilde{\mathbf{b}}_1 \cdots \tilde{\mathbf{b}}_n) e^{i\sum_j \tilde{\mathbf{a}}_j \cdot \tilde{\mathbf{x}}_j} e^{i\sum_k \tilde{\mathbf{p}}_k \cdot \tilde{\mathbf{b}}_k}$$

$$\Downarrow$$

$$P^- = \int \prod d\tilde{\mathbf{a}}_i d\tilde{\mathbf{b}}_i p^-(\tilde{\mathbf{a}}_1 \cdots \tilde{\mathbf{a}}_n, \tilde{\mathbf{b}}_1 \cdots \tilde{\mathbf{b}}_n) e^{i\sum_j \tilde{\mathbf{a}}_j \cdot \tilde{\mathbf{x}}_j} e^{i\sum_k (\tilde{\mathbf{p}}_k - e_k \tilde{\mathbf{A}}(\tilde{\mathbf{x}}_k)) \cdot \tilde{\mathbf{b}}_k}$$

The current is the coefficient of the part of this expression that is linear in $\tilde{\mathbf{A}}(\tilde{\mathbf{x}})$. For non-commuting operators $\hat{A}$ and $\hat{B}$ the linear term in $\hat{A}$ comes from differentiating the Trotter product formula and talking limits

$$\frac{d}{d\eta} e^{\hat{B}+\eta\hat{A}}_{|\eta=0} = \int_0^1 d\lambda e^{\lambda\hat{B}} \hat{A} e^{(1-\lambda)\hat{B}}.$$

Dynamical currents

To factor out the vector potential use the fact that $\tilde{\mathbf{p}}_i$ generates translations in $\tilde{\mathbf{x}}_i$

$$e^{i\lambda(\tilde{\mathbf{b}}_1\cdot\tilde{\mathbf{p}}_1+\tilde{\mathbf{b}}_2\cdot\tilde{\mathbf{p}}_2)}(-ie_1\tilde{\mathbf{A}}(\tilde{\mathbf{x}}_1,x^+)\cdot\tilde{\mathbf{b}}_1-ie_2\tilde{\mathbf{A}}(\tilde{\mathbf{x}}_2,x^+)\cdot\tilde{\mathbf{b}}_2)e_e^{i(1-\lambda)(\tilde{\mathbf{b}}_1\cdot\tilde{\mathbf{p}}_1+\tilde{\mathbf{b}}_2\cdot\tilde{\mathbf{p}}_2)}$$

The term linear in the vector potential factors out on the left $\tilde{\mathbf{A}}$

$$= (-ie_1\tilde{\mathbf{A}}(\tilde{\mathbf{x}}_1+\lambda\tilde{\mathbf{b}}_1,x^+)\cdot\tilde{\mathbf{b}}_1 - ie_2\tilde{\mathbf{A}}(\tilde{\mathbf{x}}_2+\lambda\tilde{\mathbf{b}}_2,x^+)\cdot\tilde{\mathbf{b}}_2)e^{i(\tilde{\mathbf{b}}_1\cdot\tilde{\mathbf{p}}_1+\tilde{\mathbf{b}}_2\cdot\tilde{\mathbf{p}}_2)}$$

The part of the transformed interaction linear in $\tilde{\mathbf{A}}$ becomes

$$\int\prod_i d\tilde{\mathbf{a}}_i d\tilde{\mathbf{b}}_i \int_0^1 d\lambda p_2^-(\tilde{\mathbf{a}}_1,\tilde{\mathbf{a}}_2;\tilde{\mathbf{b}}_1,\tilde{\mathbf{b}}_2)\sum(-ie_j\tilde{\mathbf{A}}(\tilde{\mathbf{x}}_j+\lambda\tilde{\mathbf{b}}_j,x^+))\cdot\tilde{\mathbf{b}}_j\times$$

$$e^{i\sum\tilde{\mathbf{a}}_m\cdot\tilde{\mathbf{x}}_m}e^{i\sum\tilde{\mathbf{b}}_k\cdot\tilde{\mathbf{p}}_k}$$

Dynamical currents

Operator structure in terms of matrix elements of interaction

$$\tilde{\mathbf{I}}_2(\tilde{\mathbf{x}},0) = -i(2\pi)^{-6}\int_0^1 d\lambda \int \prod_j d\tilde{\mathbf{a}}_j d\tilde{\mathbf{b}}_j d\tilde{\mathbf{p}}'_j \times$$

$$\langle \tilde{\mathbf{p}}'_1 + \frac{\tilde{\mathbf{a}}_1}{2}, \tilde{\mathbf{p}}'_2 + \frac{\tilde{\mathbf{a}}_2}{2} | P_2^- | \tilde{\mathbf{p}}'_1 - \frac{\tilde{\mathbf{a}}_1}{2}, \tilde{\mathbf{p}}'_2 - \frac{\tilde{\mathbf{a}}_2}{2} \rangle \times$$

$$e^{i\sum \tilde{\mathbf{a}}_i \cdot \tilde{\mathbf{X}}_i} \left(\sum e_j \delta(\tilde{\mathbf{X}}_j + \lambda \tilde{\mathbf{b}}_j - \tilde{\mathbf{x}})(i\tilde{\boldsymbol{\nabla}}_{p'_j}\right) e^{i\sum \tilde{\mathbf{b}}_j \cdot (\tilde{\mathbf{P}}_j - \tilde{\mathbf{p}}'_j)}$$

where $\tilde{\mathbf{X}}_j$ and $\tilde{\mathbf{P}}_j$ are operators.

Example - free particle - construction gives

$$\langle \tilde{\mathbf{p}}_F | \tilde{\mathbf{I}}_{lf}(\tilde{\mathbf{0}}) | \tilde{\mathbf{p}}_I \rangle =$$

$$-\frac{e}{(2\pi)^3} \int_0^1 d\lambda \left(-\frac{\lambda(\mathbf{p}_{\perp F} + (1-\lambda)\mathbf{p}_{\perp I})^2 + m^2}{(\lambda p_F^+ + (1-\lambda) p_I^+)^2}, \frac{2(\lambda \mathbf{p}_{\perp F} + (1-\lambda)\mathbf{p}_{\perp I})}{\lambda p_F^+ + (1-\lambda) p_I^+} \right)$$

Compare to the non-relativistic result

$$\langle \mathbf{p}_F | \mathbf{I}_{nr}(\mathbf{0}) | \mathbf{p}_I \rangle =$$

$$-\frac{e}{(2\pi)^3} \int_0^1 d\lambda \frac{\lambda \mathbf{p}_F + (1-\lambda)\mathbf{p}_I}{m} = -\frac{e}{(2\pi)^3} \frac{\mathbf{p}_F + \mathbf{p}_I}{2m}$$

Suprise

- **Method only gives many-body parts of bad currents,** $I_\perp(0)$ **and** $I^-(0)$**.**
- P^- **and kinematic subgroup does not fix** $\mathbf{J}_\perp$**. Missing information is needed to define a 4-vector.**
- **The good current can be extracted from the bad currents using the dynamical** $\mathbf{J}_\perp$

$$I^+(0) = I^-(0) - 2i[J^1, I^2(0)]$$

- **Requires transverse rotations (transverse rotations** $+$ **kinematic subgroup generates all generators.)**
- **Covariance is limited to the one-photon exchange approximation.**

Conclusions

- Currents in phenomenological relativistic light-front models are representation dependent.
- Current covariance and conservation require many-body parts of the current in dynamical models.
- These conditions are only constraints; they do not determine the current.
- Minimal substitution in the irreducibble kinematic representation gives a gauge invariant P^- and interaction-dependent current.
- Construction works for non-local operators but only gives the bad components of the current.
- The good component of the many-body current can be constructed using the transverse rotation generators. This fixes the meaning of covariance.

Thank you to the organizers!